Continuity

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1 Real numbers

The real numbers consist of: a set $\mathbf{R}$, together with two binary operations, formally functions $\mathbf{R} \times \mathbf{R} \to \mathbf{R}$, written $(a, b) \mapsto a + b$ and $(a, b) \mapsto ab$; two distinguished elements (i.e. elements of $\mathbf{R}$ with definite names) 0 and 1; and a binary relation written $a < b$. This set-up obeys the usual axioms (associativity etc.).

2 The real numbers as a complete ordered field

Definition 2.1 *A system satisfying the usual axioms of addition and multiplication for numbers is called an* (algebraic) field*.*

Definition 2.2 *A system satisfying the usual axioms of addition and multiplication for numbers, and the usual axioms for ordering, is called an* ordered field*.*

Completeness axiom: *Every non-empty bounded subset S of $\mathbf{R}$ has a least upper bound $\sup S$ in $\mathbf{R}$.*

Suppose now that $\mathbf{F}$ is (just known to be) an ordered field. Then the Completeness axiom makes sense in $\mathbf{F}$, by the following definitions:

Definition 2.3 *For $x, y \in \mathbf{F}$ we define $x \leq y$ to mean (either $x < y$ or $x = y$).*

Definition 2.4 *If $S \subseteq \mathbf{F}$ is a subset then $U \in \mathbf{F}$ is called an* upper bound *for S if $s \leq U \forall s \in S$.*

Definition 2.5 *If $S \subseteq \mathbf{F}$ ($S \neq \emptyset$) has an upper bound then we say that S is* bounded above*.*

Then the Completeness axiom makes sense in $\mathbf{F}$, though it may or may not be true in $\mathbf{F}$. We assume it is true for $\mathbf{R}$, so we take $\mathbf{R}$ to be a complete ordered field.

Definition 2.6 *A* least upper bound *ℓ for a nonempty subset S of any ordered field $\mathbf{F}$ is an element of $\mathbf{F}$ such that:*

1. *ℓ is an upper bound for S, and*
2. *If u is any upper bound for S then $\ell \leq u$.*

Traditionally we write $\sup S$ for the least upper bound of $S \subseteq \mathbf{R}$.
We'll assume $\mathbf{R}$ contains subsets $\mathbf{N}, \mathbf{Z}, \mathbf{Q}$ with all the usual properties.

Proposition 2.7 *If $S \subseteq \mathbf{R}$ has a least upper bound, then it has at most one.*

Proof: Suppose ℓ and ℓ' are both least upper bounds for $S \subseteq \mathbf{R}$. View ℓ as *an* upper bound and ℓ' as *least* such, and see from (2) that $\ell' \leq \ell$. Now interchange the rôles of ℓ and ℓ', and get $\ell \leq \ell'$. So $\ell = \ell'$.
Since by the axiom any non-empty $S \subseteq \mathbf{R}$ which is bounded above has a least upper bound, we now see that it is unique. Call it $\sup S$ (supremum or l.u.b.).

2.1 Remark

If $F \subseteq \mathbf{R}$ is finite (non-empty) then F contains a *greatest* member, and this is $\sup F$.

2.2 Remark

More generally (even if S isn't finite) it may be that S contains a greatest element; call it $x_{\max}$. Check:

1. For any $x \in S$, $x \leq x_{\max}$, by definition of greatest element.
2. if U is any bound for S, then (since $x_{\max} \in S$), we get $x_{\max} \leq U$.

Examples

1. In $[0, 1]$, $x_{\max} = 1$.
2. In $(0, 1)$, there's no greatest element.
3. In $\{1 - \frac{1}{n} : n \in \mathbf{N}_+\} = \{0, \frac{1}{2}, \frac{2}{3}, \cdots\}$, there's no greatest element.

NB: Not all subsets of $\mathbf{R}$ are intervals.

2.3 Remark

If $S \subseteq \mathbf{R}$ is non-empty and bounded above but has no greatest element, than $\sup S$ is a kind of substitute for the greatest element $x_{\max}$.

Proposition 2.8 (The Archimedian Property) .

(a) For any $x \in \mathbf{R}$, $\exists n \in \mathbf{N}$ such that $n > x$.

(b) For any $x > 0$ in $\mathbf{R}$, $\exists n \in \mathbf{N}$ such that $\frac{1}{n} < x$.

Proof

(a) Suppose there is some $x \in \mathbf{R}$ for which (a) fails, so $n \leq x \; \forall n \in \mathbf{N}$.
So $\mathbf{N}$ is non-empty and bounded above. By the completeness axiom, $\mathbf{N}$ has a supremum, say ℓ.
For any $n \in \mathbf{N}$, we have $n + 1 \in \mathbf{N}$, so $n + 1 \leq \ell$. Hence $n < \ell - 1$.
But this is true for any $n \in \mathbf{N}$, so $\ell - 1$ is another upper bound for $\mathbf{N}$, so ℓ is not the least such. This contradicts the choice of ℓ, so (a) must be true.

(b) If $x > 0$ then $\frac{1}{x} > 0$, so by (a) $\exists n \in \mathbf{N}$ with $n > \frac{1}{x}$, so $\frac{1}{n} < x$.

2.4 Remark

For some subsets of $\mathbf{R}$, even when there's no greatest element, we can see that a supremum exists without having to use the completeness axiom.

Example

To prove that $\sup(0,1) = 1$: let $S = (0,1)$.

1. $x \in S \Rightarrow x < 1$, by definition of $(0,1)$.
2. Let U be any upper bound for S. We need to show that $1 \leq U$. Suppose for a contradiction that $U < 1$. So $1 - U > 0$. So by 1.8(b), $\exists n \in \mathbf{N}$ such that $1 - U > \frac{1}{n}$. So $1 - \frac{1}{n} > U$. But $1 - \frac{1}{n} \in (0,1)$ (NB: we can w.l.o.g.[1] assume that $n > 1$). So U is not an upper bound for S.

2.5 Remark

We now look at an example where we *do* need the axiom.

Consider $S = \{x \in \mathbf{R} : x^2 < 2\}$. Then $S \neq \emptyset$ (e.g. $1 \in S$) and S is bounded above, e.g. by 2, since if $x > 2$ then $x^2 > 4$, so $x \notin S$.

By completeness, S has a supremum, say ℓ. We'll show that $\ell^2 = 2$, i.e. $\ell = \sqrt{2}$.

Proof (by contradiction)

1. Suppose $\ell^2 < 2$. We'll show that ℓ is not an upper bound for S, by showing that $\exists n \in \mathbf{N}$ with $\ell + \frac{1}{n} \in S$.

   We want $\left(\ell + \dfrac{1}{n}\right)^2 < 2$, i.e. $\ell^2 + \dfrac{2\ell}{n} + \dfrac{1}{n^2} < 2$.

   Equivalently $\dfrac{2\ell}{n} + \dfrac{1}{n^2} < 2 - \ell^2$ (NB: $2 - \ell^2 > 0$ by assumption).

   Well, $\dfrac{2\ell}{n} + \dfrac{1}{n^2} \leq \dfrac{2\ell}{n} + \dfrac{n}{n^2}$, so enough to get n such that $\dfrac{2\ell + 1}{n} < 2 - \ell^2$.

   But by 1.8(a), $\exists n \in \mathbf{N}$ with $n > \dfrac{2\ell + 1}{2 - \ell^2}$.

2. Suppose $\ell^2 > 2$. We'll show that ℓ isn't "least" by finding n such that $\ell - \frac{1}{n}$ is an upper bound of S.

   Enough to see $\left(\ell - \dfrac{1}{n}\right)^2 > 2$ for then if $x \in S$, we have:

   $x^2 < 2 < \left(\ell - \dfrac{1}{n}\right)^2$ and we get $x < \ell - \dfrac{1}{n}$.

   (NB: $\ell - \dfrac{1}{n} > 0$ since $1 \in S$, so $\ell \geq 1$)

   We want $\ell^2 - \dfrac{2\ell}{n} + \dfrac{1}{n^2} > 2$. Enough to get $\ell^2 - \dfrac{2\ell}{n} > 2$

[1] without loss of generality

, or equivalently $\frac{2\ell}{n} < \ell^2 - 2$. Just choose $n > \frac{2\ell}{\ell^2 - 2}$.

Remark

If $S = \emptyset$, then vacuously any real number is an upper bound.

Proposition 2.9 *Let S be a non-empty subset of $\mathbf{R}$ which is bounded above. The $\ell = \sup S$ iff:*

1. *ℓ is an upper bound for S.*
2. *given any (real) $\epsilon > 0$, $\exists s \in S$ with $s > \ell - \epsilon$.*

Proof Suppose first $\ell = \sup S$. Then (1) is true by definition. Now for any $\epsilon > 0$, since ℓ is the least upper bound, $\ell - \epsilon$ is not an upper bound for S. So $\exists s \in S$ such that $s > \ell - \epsilon$.

Conversely, supppose (1) and (2) hold. Then by (1) ℓ is an upper bound. We want to see that it's the least such. So let U be any upper bound for S. We want $U \geq \ell$. Suppose for a contradiction $U < \ell$. Put $\epsilon = \ell - U$ in (2). So $\exists s \in S$ with $s > \ell - \epsilon = \ell - (\ell - U) = U$. So U is not an upper bound.

Proposition 2.10 *Suppose $S, T \subseteq \mathbf{R}$, non-empty and bounded above. Then $S \cup T$ is bounded above and $\sup(S \cup T) = \max(\sup S, \sup T)$.*

Proof: exercise.

Proposition 2.11 *Suppose S, T as in 2.10. Let $S + T = \{x + y \in \mathbf{R} : x \in S, y \in T\}$. Then $S + T$ is bounded and $\sup(S + T) = \sup S + \sup T$.*

Proof For any $x \in S$, $y \in T$ we have $x \leq \sup S$, $y \leq \sup T \Rightarrow \sup S + \sup T$ is an upper bound for $S + T$.

Let $\epsilon > 0$. BY 2.9, $\exists s \in S$ such that $s > \sup S - \frac{\epsilon}{2}$ and $\exists t \in T$ such that $t > \sup T - \frac{\epsilon}{2}$. Then $s + t \in S + T$, and $s + t > \sup S + \sup T - \epsilon$. So by 2.9, $\sup S + \sup T = \sup(S + T)$.

Warning: Given $\ell = \sup S$ and $\epsilon > 0$, it's not necessarily true that $\ell - \epsilon \in S$. All you can be sure of is that $\exists s \in S$ with $s > \ell - \epsilon$.

Example: $S = \{1 - \frac{1}{n} : n \in \mathbf{N}_+\}$. Then $\sup S = 1$, and $\epsilon = 3/10 > 0$, $\ell - \epsilon = 7/10$, not in S, but for example $9/10 \in S$.

Definition 2.12 *Given $S \subseteq \mathbf{R}$, non-empty and bounded below, a greatest lower bound G for S is a number such that:*

1. *g is a lower bound of S.*
2. *g is the greatest such.*

Written $g = \inf S$ (infimum).

Proposition 2.13 *If S is non-empty, $S \subseteq \mathbf{R}$, and bounded below, then $\exists \inf S$.*

Proof follows from Completeness Axiom by a trick: let $S^* = \{x \in \mathbf{R} : -x \in S\}$. Easy to check: x is a lower bound for S iff $-x$ is an upper bound for S^*.

Outline: Since S is bounded below, S^* is bounded above and non-empty, so by Completeness $\sup S^*$ exists. Not hard to check that $-\sup S^*$ is a greatest lower bound for S.

3 Revision on limits of functions

Definition 3.1 *Suppose a real-valued function f is defined at least 'near' a, i.e. $\exists(b,c)$ with $a \in (b,c)$ and f defined at least on $(b,c)\backslash\{a\}$. Then $\lim_{x\to a} f(x) = \ell$ means: Given any $\epsilon > 0$, $\exists\delta > 0$ such that whenever $0 < |x-a| < \delta$, we have $|f(x) - \ell| < \epsilon$.*

N.B. For $\lim_{x\to a} f(x)$ to exist, it's not necessary for f to be defined at a, and if $f(a)$ is defined it need not be equal to $\lim_{x\to a} f(x)$.

Example

Suppose $f : [0,1] \to \mathbf{R}$ is defined by

$$f(x) = \begin{cases} x & : \quad x \neq 1/2 \\ 666 & : \quad x = 1/2 \end{cases}$$

Then $\lim_{x\to 1/2} f(x) = 1/2 \neq f(1/2)$. (Or: don't define $f(1/2)$.

Example

$$\text{Let } f(x) = \begin{cases} \sin(1/x) & : \quad x \neq 0 \\ 0 & : \quad x = 0 \end{cases}$$

Want to show that $\lim_{x\to 0} f(x)$ does not exist.

Three levels:

1. (intuitive) The graph osciallates too much near 0 for $\lim_{x\to 0} f(x)$ to exist.
2. (towards proof) Take any potential limit ℓ. Then near 0, there are points where the graph is at $+1$ and points where it's at -1. These can't both be arbitrarily close to ℓ.
3. Let $\ell \in \mathbf{R}$ and take $\epsilon = 1/2$, say. Want to show that 3.1 doesn't work for this f, ϵ, ℓ.

   Take any $\delta > 0$. Let $N \in \mathbf{Z}$, $N > 1/\delta$ and consider

   $$x_1 = \frac{1}{(2N + \frac{1}{2})\pi}, \; x_2 = \frac{1}{(2N + \frac{3}{2})\pi}.$$

   We see $f(x_1) = 1$, $f(x_2) = -1$, and $0 < |x_1 - 0| < \delta$. Now if $0 < |f(x) - \ell| < 1/2$ whenever $0 < |x - 0| < \delta$, we'd have $|1 - \ell < 1/2$, $| - 1 - \ell < 1/2$ so
   $2 = |1 - (-1)| = |1 - \ell + -1 - \ell| \leq |1 - \ell| + | - 1 - \ell| < 1$. Contradiction.

One-sided limits

If f is defined at least on (a,b) $(b > a)$ we say $\lim_{x\to a+} f(x) = \ell$ or $f(x) \to \ell$ as $x \to a+$ if given any $\epsilon > 0$, $\exists\delta > 0$ such that $|f(x) - \ell| < \epsilon$ whenever $a < x < x + \delta$.

Similarly for $\lim_{x\to b-} f(x)$, $\lim_{x\to\infty} f(x)$, $\lim_{x\to-\infty} f(x)$.

Definition 3.2 *Suppose $f : (b,c) \to \mathbf{R}$ and $a \in (b,c)$. We say f is continuous at a if $f(x) \to f(a)$ as $x \to a$.*

Variants: $f : [a, b] \to \mathbf{R}$ is (right-hand) continuous at a is $f(x) \to f(a)$ as $x \to a+$. Similarly for left-hand continuity at b.

Proposition 3.3 *$f : (b, c) \to \mathbf{R}$ is continuous at $a \in (b, c)$ iff given any $\epsilon > 0$, $\exists\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$.*

Proof follows from definitions 3.1 and 3.2.

Note Difference from 3.1: no "$0 < |x - a|$ this time. When $x = a$, $f(x) = f(a)$ so $|f(x) - f(a)| = 0 < \epsilon$, i.e. this time we're looking at $f(x) \to f(a)$ as $x \to a$, not $f(x) \to \ell$ as $x \to a$.

There are two ways of thinking about continuity of f at a:

1. Approximation: think of f as a black box which accepts real number inputs and gives you real number outputs. Then if you vary input x only slightly from a, output $f(x)$ also varies only slightly from $f(a)$.

2. Geometrically: graph doesn't jump around too much: given any horizontal strip centred on height $f(a)$ you can find a vertical strip centred on a, narrow enough so that all of the graph in the vertical strip is also inside the horizontal strip. For some reason he called this the "principle of inertia".

Definition 3.4 *$f : [a, b] \to \mathbf{R}$ is said to be continuous if it's continuous at every point in $[a, b]$ (RH continuous at a, LH continuous at b).*

Note if $f : [a, b] \to \mathbf{R}$ is continuous and I is any interval $I \subseteq [a, b]$, then $f|_I : I \to \mathbf{R}$ is also continuous (obvious if you ask me). For $c \in I$, any $\epsilon > 0$, by continuity of f at c, $\exists\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$ and $x \in [a, b]$ so certainly true $\forall x$ with $|x - c| < \delta$ and $x \in I$.

Continuous functions behave well under addition and multiplication.

Proposition 3.5 *Suppose $f, g : (b, c) \to \mathbf{R}$ and $a \in (b, c)$ and f, g continuous at a. Then so are the following functions:*

1. *$x \mapsto kf(x)$ (k constant)*

2. *$x \mapsto f(x) + g(x)$ ("$f + g$")*

3. *$x \mapsto f(x)g(x)$*

4. *provided $g(a) \neq 0$, $x \mapsto 1/g(x)$*

Proof: Given definition 3.2, follows imediately from Algebra of Limits.

Corrolary 3.6 *Any polynomial $p(x)$, $x \mapsto a_0 + a_1x + \ldots + a_nx^n$, a_i constant, is continuous at any point in $\mathbf{R}$ and so is any rational function $x \mapsto \frac{p(x)}{q(x)}$ where p and q are polynomials, except at (finite set of) points where $q(x) = 0$.*

Proof A sequence of easy inductions. But first,

1. Any constant function $x \mapsto k$ is continuous, since for any $\epsilon > 0$ take any $\delta > 0$ and we have $|k - k| = 0 < \epsilon$ whenever $|x - a| < \delta$.

2. The function $x \mapsto x$ is continous, since for any $\epsilon > 0$, take $\delta = \epsilon$. Then $|x - a| < \delta \Rightarrow |f(x) - f(a)| = |x - a| < \epsilon$.

First induction: $\forall n \in \mathbf{N},\ x \mapsto x^n$ is continuous. Anchor is (2) above. Induction step follows from $x \mapsto x$ and $x \mapsto x^n$ continuous $\Rightarrow x \mapsto x^{n+1}$ continuous.

Then by one more application of 'product', $x \mapsto a_n x^n$ is continuous. Finally $x \mapsto a_0 + a_1 x + \ldots + a_n x^n$ is continuous by easy induction on adding two continous functions.

Also, $p(x)/q(x)$ follows by "quotient" part of Algebra of Limits.

Remark Continuity of f at a can be split into four bits:

1. $\lim_{x \to a+}$ exists.

2. $\lim_{x \to a+} = f(a)$

3. $\lim_{x \to a-}$ exists.

4. $\lim_{x \to a-} = f(a)$

Examples

$$f(x) = \begin{cases} 0 & : \quad x \le 0 \\ 1 & : \quad x > 0 \end{cases}, \ a = 0.\ 1,\ 3,\ 4 \text{ true but } 2 \text{ fails at } 0.$$

$$f(x) = \begin{cases} 0 & : \quad x < 0 \\ 1 & : \quad x \ge 0 \end{cases}, \ a = 0.\ 1,\ 2,\ 3 \text{ hold but } 4 \text{ fails at } 0.$$

These two functions have a *simple jump discontinuity.*

$$f(x) = \begin{cases} \sin(1/x) & : \quad x > 0 \\ 0 & : \quad x \le 0 \end{cases}, \ a = 0.\ 3,\ 4 \text{ true, } 1 \text{ fails so } 2 \text{ meaningless.}$$

Similarly we can get 1 and 2 to hold and 3 to fail so 4 is meaningless.

Proposition 3.7 *The composition $f \circ g$ of continuous functions is continuous.*

Proposition 3.8 *If f is differentiable at $a \in \mathbf{R}$, it's continuous at a.*

$$\frac{f(x) - f(a)}{x - a} \to f'(a) \text{ as } x \to a \Rightarrow \underbrace{f(x) - f(a)}_{\text{hence also} \to 0} - \underbrace{(x - a)f'(a)}_{\text{must tend to 0}} \to 0 \text{ as } x \to a$$

4 Results about sequences proved using Completeness, and needed to study Continuity

Proposition 4.1 *Any monotone bounded sequence converges.*

Proof First consider (a_n) increasing, bounded above. Let $S = \{a_n : n \in \mathbf{N}\}$. Certainly non-empty, and bounded above. So the least upper bound $\sup S$ exists — call it ℓ. We'll show (a_n) converges to ℓ.

Let $\epsilon > 0$. The $\ell - \epsilon$ is not an upper bound for S. So $\exists N$ such that $a_N < \ell - \epsilon$. By an easy induction using the fact $a_{n+1} \geq a_n \forall n$, get $a_n > \ell - \epsilon \forall n \geq N$. But also $a_n \leq \ell \forall n$. So $\forall n \geq N$ we have $\ell - \epsilon < a_n \leq \ell$, so $|a_n - \ell| < \epsilon$.

Finally, if (a_n) is decreasing and bounded below, then $(-a_n)$ is increasing and bounded above, so $-a_n \to \ell$, say, as $n \to \infty$, so it's easily seen that $a_n \to -\ell$ as $n \to \infty$.

Theorem 4.2 (Bolzano-Weierstrass) *Any bounded sequence of real numbers has at least one convergent subsequence.*

Proof I It is enough to show that (a_n) has a monotonic subsequence. For then that subsequence is monotonic and bounded so 4.1 applies. We can show ...

Theorem 4.3 *Any sequence of real numbers has a monotonic subsequence.*

Local definition: for a sequence (a_n), n is a scenic viewpoint if $\forall m \geq n$, $a_n \geq m$.

Proof The sequence (a_n) has infinitely many scenic viewpoints, or it doesn't.

Case 1 Suppose that there are infinitely many scenic viewpoints. Put them in order: $n_1 < n_2 < n_3 < \ldots$. Consider a_{n_r}: since n_r is a scenic viewpoint, $a_m \leq a_{n_r} \forall m \geq n_r$, in particular $a_{n_{r+1}} \leq a_{n_r}$. This is true $\forall r$ so (a_{n_r}) is decreasing.

Case 2 There are not infinitely many scenic viewpoints. Let n_0 be the largest scenic viewpoint, if any exist, otherwise let $n_0 = 0$. Let $n_1 = n_0 + 1$. Then n_1 is not a scenic viewpoints so $\exists m > n_1$ such that $a_m > a_{n_1}$. Choose any such m and call it n_2. Inductively: suppose $n_1 < n_2 < \ldots < n_r$ chosen so that $a_{n_1} < a_{n_2} < \ldots < a_{n_r}$. Then n_r is not a scenic viewpoint, so we can choose $n_{r+1} > n_r$ with $a_{n_{r+1}} > a_{n_r}$. This way, inductively, we get a (strictly) increasing subsequence of (a_n).

Proof II (Bisection method)

Definition: call an interval I *crowded* if $a_n \in I$ for infinitely many I.

Step 1. (a_n) is bounded so $\exists K \in \mathbf{R}$ such that $a_n \in [-K, K] \forall n$. So either $\dagger[-K, 0]$ is crowded or $[0, K]$ is crowded. If †holds, put $x_0 = -K$, $y_0 = 0$. If †fails, put $x_0 = 0, y_0 = K$. Then $[x_0, y_0]$ is crowded.

Suppose inductively we've already chosen $x_0 \leq x_1 \leq \ldots \leq x_r \leq y_r \leq y_{r-1} \leq \ldots \leq y_0$ such that $[x_r, y_r]$ is crowded and $y_i - x_i = K/2^i$ for $i = 0, 1, \ldots, r$.

Bisect $[x_r, y_r]$. Either ‡the left-hand half or the right-hand half is crowded. If ‡holds, put $x_{r+1} = x_r$, $y_{r+1} = (x_r + y_r)/2$. Otherwise put $x_{r+1} = (x_r + y_r)/2$, $y_{r+1} = y_r$. Then clearly the assumption hold for $n + 1$.

Not (x_r) in increasing and bounded above (by any y_s) so $x_r \to x$ as $r \to \infty$ and similarly $y_r \to y$ as $r \to \infty$ and at any stage r we have $x_r \leq x \leq y \leq y_r$ ($y_s \geq x$ for any s since $x_r \leq y_s$ (fixed) $\forall r$). Put $y_r - x_r = K/2^r$ — true $\forall r$ so by sandwiching, $y = x$.

Now need to see that a subsequence of (a_n) converges to x. Every $[x_r, y_r]$ is crowded so $a_{n_r} \in [x_r, y_r]$ for some n_r. But this is not yet an orderly subsequence ($n_{r+1} > n_r \forall r$). Suppose we already have $n_1 < n_2 < \ldots < n_r$ such that $a_r \in [x_i, y_i] \forall i \leq r$. Since $[x_{r+1}, y_{r+1}]$ is crowded, we can choose $n_{r+1} > n_r$ such that $a_{n_{r+1}} \in [x_{r+1}, y_{r+1}]$. Now we see $x_r \leq a_{n_r} \leq y_r \forall r$, so $a_{n_r} \to x$ as $r \to \infty$, by sandwiching.

There's another version of B-W — see e.g. Spivak *Calculus* p. 386 ex. 21.

5 Results about continuous functions $f : [a, b] \to \mathbf{R}$

Theorem 5.1 *Any continuous function $f : [a, b] \to \mathbf{R}$ is bounded and attains its bounds.*

We now rephrase the second part of this assertion thrice.

Suppose $f : [a, b] \to \mathbf{R}$ is bounded, so $\exists K$ such that $|f(x)| \leq K \forall x \in [a, b]$. Equivalently, $f([a, b]) = \{y \in \mathbf{R}$ such that, for some $x \in [a, b], f(x) = y\} \subseteq [-K, K]$.

So image $f([a, b])$ is (non-empty) bounded above and below, so has a sup and an inf; (ii) says that they are taken on as values of f. Equivabloodylently, $\exists c, d \in [a, b]$ s.t. $f(c) \leq f(x) \leq f(d) \forall x \in [a, b]$.

Remark It's crucial that [a,b] is closed and bounded.

Examples

- Define $f : (0, 1) \to \mathbf{R}$ by $f(x) = 1/x$. f continuous, not bounded.
- Define $g : (0, 1) \to \mathbf{R}$ by $g(x) = x$. Continuous, bounded but bounds not attained.
- Define $g : [1, \infty) \to \mathbf{R}$ by $g(x) = x$. Not bounded.
- Define $f : [1, \infty) \to \mathbf{R}$ by $= 1/x$. Bounded but infimum not attained.

Recall If (a_n) converges to ℓ and $a_n \geq L \forall n$, then $\ell \geq L$ (also, $a_n > L \Rightarrow \ell \geq L$, not $\ell > L$). Similarly, $a_n \leq L \forall n \Rightarrow \ell \leq L$.

5.0.1 Proof

(i) Suppose f not bounded. Then for any $n \in \mathbf{N}$, $\exists a_n \in [a, b]$ with $|f(a_n)| > n$. Consider (a_n) — bounded, so by Bolzano-Weierstrass, has a convergent subsequence a_{n_r} converging to some $\ell \in \mathbf{R}$ †.

Now $a \leq a_{n_r} \leq b \forall r$, so $a \leq \ell \leq b$, i.e. $\ell \in [a, b]$ ‡. Use continuity of f at ℓ: get $f(a_{n_r}) \to f(\ell)$ as $r \to \infty$. All we need of this is: $(f(a_{n_r}))$ converges, hence (from result last term) is bounded. But $|f(a_{n_r})| > n_r \forall r$. Contradiction, so f bounded.

†is where we use $[a, b]$ bounded. ‡is where we use $[a, b]$ closed.

(ii) Let g, ℓ be the inf and sup of $f([a, b])$. Suppose e.g. ℓ is not attained on $[a, b]$. We know $f(x) \leq \ell \forall x \in [a, b]$ and by our supposition $f(x) < \ell \forall x \in [a, b]$. **KEY:** Let $h(x) = 1/(\ell - f(x))$, $x \in [a, b]$. By (3.5), h is continuous on $[a, b]$. By (1), h is bounded on $[a, b]$. Say

$1/(\ell - f(x)) \leq U \Rightarrow \ell - f(x) \geq 1/U \Rightarrow f(x) \leq \ell - 1/U \forall x \in [a,b]$. i.e. $\ell - 1/U$ is an upper bound for $f([a,b])$ — contradicts leastness of ℓ. Proof that $\inf f([a,b])$ is attained on $[a,b]$ is similar.

Theorem 5.2 (Special case of IVT) *Suppose $f : [a,b] \to \mathbf{R}$ is continuous, and $f(a) < 0$ and $f(b) > 0$. Then $\exists c \in (a,b)$ with $f(c) = 0$.*

Note not true for $\mathbf{Q}$: try $f : [0,2] \to \mathbf{R}$, $f(x) = x^2 - 2$.

Proof I

Consider $s = \{x \in [a,b] : f(x) \leq 0\}$. Note $s \neq 0$ ($a \in S$) and bounded above (e.g. by b). So by Completeness, $\exists \sup S$. Let $c = \sup S$. We'll show $f(c) = 0$ by contradicting the other possibilities.

1. Suppose $f(c) > 0$. We'll show this contradicts c being *least* upper bound of S. By continuity of f at c, $\exists \delta > 0$ such that whenever $x \in [a,b]$ and $|x - c| < \delta$, we have $|f(x) - f(c)| < f(c)/2$ (note $f(c)/2 > 0$).

   For all such x, $-(f(x) - f(c)) < f(c)/2$, so $f(x) > f(c) - f(c)/2 > 0$. Now $c \neq 0$ (we're supposing $f(c) > 0$, and we know $f(a) < 0$). We may as well assume $\delta < c - a$, so $c - \delta > a$. So $\forall x \in (c - \delta, c]$¡ we have $f(x) > 0$. By definition of S, $f(x) > 0 \forall x > c$ (in $[a,b]$). So $c - \delta$ is an upper bound for S, contradicting leastness of c.

2. Suppose $f(c) < 0$. We'll contradict "c is *an* upper bound for S". By continuity of f at c, $\exists \delta > 0$ such that whenever $x \in [a,b]$ and $|x - c| < \delta$, we have $|f(x) - f(c)| < -f(c)/2$ (note $-f(c)/2 > 0$).

   So for all such x we have $f(x) - f(c) < -f(c)/2$ so $f(x) < f(c) - f(c)/2 < 0$. Now $c \neq b$ ($f(c) < 0$, $f(b) > 0$). So certainly $\exists x \in [a,b]$ with $c < x < x + \delta$, and for any one such, $f(x) < 0$, so contradicting "c an upper bound for S".

So $f(c) = 0$.

Proof II

Proceed similarly to Bolzano-Weierstrass proof, but choose $a = x_0 \leq x_1 \leq \ldots \leq x_r \leq y_r \leq y_{r-1} \leq \ldots \leq y_0 \leq b$ as follows:

If $f((a+b)/2) \leq 0$, put $x_1 = (a+b)/2$, $y_1 = b$.

If $f((a+b)/2) > 0$, put $x_1 = a$, $y_1 = (a+b)/2$.

Continue inductively — get $a \leq x_1 \leq \ldots \leq x_r \leq y_r \leq y_{r-1} \leq \ldots \leq y_1 \leq b$. such that $y_r - x_r = (b-a)/2^r$ and $f(x_r) \leq 0$, $f(y_r) \geq 0$. As in Bolzano-Weierstrass, $(x_r), (y_r)$ converge to a common limit c and $f(c) \leq 0$ by continuity of f at c, and $f(c) \geq 0$ ($f(x_r) \leq 0 \forall r$) so $f(c) = 0$.