

Continuity summary

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- 1 Real numbers as ordered set with multiplication and addition
- 2 \mathbf{R} as complete ordered field
 - 2.1 Definition of algebraic field
 - 2.2 Definition of ordered field. Completeness axiom
 - 2.3 Definition of \leq
 - 2.4 Definition of upper bound
 - 2.5 Definition of “bounded above”
 - 2.6 Definition of supremum
 - 2.7 Uniqueness of supremum
 - 2.8 Archimedean property. Use of completeness to prove existence of $\sqrt{2}$
 - 2.9 Another definition of supremum, in terms of ϵ
 - 2.10 Supremum of unions of sets
 - 2.11 Supremum of sums of sets
 - 2.12 Definition of infimum
 - 2.13 Proof of existence of infimum on bounded-below sets, by completeness
- 3 Revision on limits of functions
 - 3.1 Definition of limits of functions. One-sided limits
 - 3.2 Definition of continuity at a point
 - 3.3 Same again
 - 3.4 Definition of continuity on an interval
 - 3.5 Well-behavedness of continuity under addition etc.
 - 3.6 Continuity of polynomials
 - 3.7 Continuity under composition
 - 3.8 Differentiability implies continuity
- 4 Results about sequences proved using completeness, and needed to study continuity
 - 4.1 Any monotone bounded sequence converges
 - 4.2 Bolzano-Weierstrass theorem (two proofs)

- 4.3 Any sequence of \mathbf{R} has a monotonic subsequence
- 5 Results about continuous functions on a bounded interval
 - 5.1 Any CFOABI is bounded and attains its bounds
 - 5.2 Special case of IVT (two proofs)
 - 5.3 IVT
 - 5.4 Corollary: existence of positive roots of positive reals
 - 5.5 Corollary: any odd real polynomial has ≥ 1 real root
 - 5.6 Corollary: continuous image of a CBI is a CBI
 - 5.7 Existence of a fixed point of $f : [a, b] \rightarrow [a, b]$
- 6 An inverse function theorem
- 7 Intervals in \mathbf{R}
 - 7.1 Definition of intervals in terms of betweenness
- 8 Integration
 - 8.1 Definition of step function
 - 8.2 Definition of partition
 - 8.3 Definition of integral of step functions
 - 8.4 Equality of integrals of subordinate partitions of a step function
 - 8.5 Definition of refinement
 - 8.6 Existence of common refinement of partitions; independence of integral from partition used
 - 8.7 Set of step functions as a vector space; integration as a linear function from this space to \mathbf{R}
 - 8.8 Positivity
 - 8.9 Biggerness of integral of a bigger function. Indicator functions
- 9 Integrals of continuous functions
 - 9.1 Integral as supremum of set of all integrals of step functions
 - 9.2 Proof
 - 9.3 Corollary: Integral lies between smallest and biggest rectangles
 - 9.4 Corollary: Integral has same area as some $f(\xi)$ -height rectangle
 - 9.5 How not to integrate
 - 9.6 Cobbling step functions together
 - 9.7 Integral of sum of step functions is sum of integrals
 - 9.8 Similarly for integrals
 - 9.9 Definition of upside-down integral
 - 9.10 Corollary to 9.8: addition of integrals in any order

- 10 Fundamental theorem of calculus
 - 10.1 FTC mk. I (on indefinite integrals)
 - 10.2 Definition of an anti-derivative (primitive)
 - 10.3 Constant difference of anti-derivatives
 - 10.4 FTC mk. II (on anti-derivative)
- 11 Applications of FTC
 - 11.1 Linearity of integral on continuous functions
 - 11.2 Corollary: bigness of integrals of bigger functions
 - 11.3 Integration by substitution
 - 11.4 Misuse and correct use of substitution
 - 11.5 Integration by parts
 - 11.6 Alternative definition of log
 - 11.7 Equivalence of definitions of log
- 12 Taylor's theorem revisited
 - 12.1 Taylor's theorem in integral form
 - 12.2 Example: power series expansion of $\log(1+x)$
- 13 Second MVT for integrals
- 14 Numerical integration; error analysis
 - 14.1 Maximum error of trapezium integration
 - 14.2 Proof
- 15 Metric spaces
 - 15.1 Definition of continuity for any $f : \mathbf{R}^n \rightarrow \mathbf{R}$
 - 15.2 Definition of a metric space
 - 15.3 Continuity at a point for any function between metric spaces
 - 15.4 Likewise over subset of space
 - 15.5 Definition of open spherical neighbourhood
 - 15.6 Translation
 - 15.7 Definition of an open set
 - 15.8 Openness of open spherical neighbourhoods
 - 15.9 Warning: dependence of openness on subspace
 - 15.10 Continuity in terms of open sets
 - 15.11 Openness of intersections and unions of open sets
 - 15.12 Topological equivalence of metrics
 - 15.13 Transitivity of openness

- 15.14 Definition of closedness
- 15.15 Warning: a set can be neither open nor closed (or both)
- 15.16 Definition of a limit point
- 15.17 Definition of closure
- 16 Compact spaces
 - 16.1 Definition of cover and subcover
 - 16.2 Definition of compactness
 - 16.3 Heine-Borel theorem (compactness of closed intervals)
 - 16.4 Boundedness and closedness of compact subspaces
 - 16.5 Converse
 - 16.6 Compactness of continuous images of compact sets
 - 16.7 Corollary: Boundedness of a continuous function on a compact space
- 17 Connected sets
 - 17.1 Definition of connectedness and disconnectedness
 - 17.2 Partition of a metric space; connectedness as unpartitionability
 - 17.3 Equivalence of definitions
 - 17.4 Connectedness of open intervals in \mathbf{R}
 - 17.5 Connectedness of continuous image of connected space
 - 17.6 The image of a \mathbf{R} -valued continuous function on a connected space is an interval.